THE CORRECTION OF CURRENT LIFTING ROTOR THEORY FOR THE EFFECTS OF THE RADIAL VELOCITY COMPONENT

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Approved:

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LIST OF SYMBOLS

α' rotor blade element angle of attack in true flow plane

α_R rotor angle of attack (positive below TPP and $\approx \tan^{-1} D_F/W$)

δ_o constant part of profile drag coefficient (≈ 0.008)

ϵ variable part of profile drag coefficient (≈ 0.008)

Γ local blade circulation

λ non-dimensional component of velocity perpendicular to TPP ,

$$\lambda = \frac{V \sin \alpha_R - v}{\Omega R}$$

η angle between true flow plane and plane perpendicular to blade axis

μ helicopter advance ratio or forward speed ratio, $\mu = \frac{V \cos \alpha_R}{\Omega R}$

Ω rotor angular velocity

Φ_n integral multiplier in equation for lift power coefficient,

$$\Phi_n = \int_{x_1}^{x_2} \frac{x^{n-1}\, dx}{(x^2 + \mu^2)^{3/2}}$$

Φ'_n integral multiplier in equation for profile power coefficient,

$$\Phi'_n = \int_{x_1}^{x_2} \frac{x^{n-1}\, dx}{\left[x^2 + (\mu^2 + \lambda^2)\right]^{3/2}}$$

ρ air density

ϕ inflow angle in plane perpendicular to blade axis

ϕ' inflow angle in true flow plane

ψ blade angular or azimuth position

σ rotor solidity, $\sigma = bc/\pi R$

θ blade element angle in plane perpendicular to blade axis, $\theta = A_o + \theta_1 x - a_1 \sin\psi + b_1 \cos\psi$

θ_1 blade twist (negative for washout)

θ' blade element angle in true flow plane

a_o rotor coning angle

A_o rotor collective pitch used to control thrust, rotor rpm, normal acceleration, etc.

a_1 rotor blade lateral cyclic pitch (furnishes normal acceleration control at high speed)

a slope of blade element lift curve ($\approx 2\pi$)

b_1 rotor blade longitudinal cyclic pitch (furnished bank angle control)

b number of blades in rotor

c' blade chord in true flow plane

c blade chord in plane perpendicular to blade axis

C_{do} local blade element profile drag coefficient, $C_{do} = 2\,C_f(1 + C_l^2)$

C_f skin friction drag coefficient (≈ 0.004)

C_l local blade element lift coefficient

C_{in} coefficient in box of the i-th column and the n-th row of power coefficient equations

C_{MT} non-dimensional average blade thrust moment coefficient, $C_{MT} = \dfrac{M_T}{\rho \pi \Omega^2 R^5}$

C_{MX} non-dimensional rotor rolling moment coefficient, $C_{MX} = \dfrac{M_X}{\rho \pi \Omega^2 R^5}$

C_{MY} non-dimensional rotor pitching moment coefficient, $C_{MY} = \dfrac{M_Y}{\rho \pi \Omega^2 R^5}$

C_P power coefficient, $C_P = \dfrac{P}{\rho \pi \Omega^3 R^5}$

C_{PR} helicopter power coefficient neglecting radial component of velocity

C_{PRL} lift power coefficient with radial component

C_{PRD} profile power coefficient with radial component

C_{PRT} helicopter power coefficient with radial component, $C_{PRT} = C_{PRL} + C_{PRD}$

C_T average rotor thrust coefficient, $C_T = \frac{T}{\rho \pi \Omega^2 R^4}$

C'_T rotor blade thrust force coefficient

C_X rotor X force coefficient, $C_X = F_X / \frac{1}{2} \rho \pi \Omega^2 R^4$

dD represents the resultant blade element differential profile drag force

$(dD)_a$ axial component of differential blade element profile drag force

$(dD)_r$ radial component of differential blade element profile drag force

$(dD)_t$ tangential component of differential blade element profile drag force

D_F fuselage parasite drag force, $D_F = f \frac{1}{2} \rho V^2$

F variable term in binomial expansion, $F = \frac{\mu x \sin \gamma}{x^2 + \mu^2}$

F_X rotor X-force

f fuselage equivalent parasite area

G variable term in binomial expansion, $G \approx \frac{\mu x \sin \gamma}{x^2 + \mu^2 + \lambda^2}$

I_1 rotor blade mass moment of inertia

K rotor blade inertia factor, $K = \frac{\rho \pi R^5 \sigma a}{2bI_1}$

dL represents the blade element resultant differential lift force

$(dL)_a$ axial component of differential blade element lift force

$(dL)_t$ tangential component of differential blade element lift force

M equilibrium factor, term in a_1 or A_o equation (see Appendix)

M_T rotor blade thrust moment

M_X rotor rolling moment

M_Y rotor pitching moment

N equilibrium factor, term in a_1 or A_o equation (see Appendix)

P power

P_R total rotor shaft horsepower required, $P_R = (C_{PRL} + C_{PRD})\rho\pi\Omega^3R^5/550$ or $P_R = C_{PR}\rho\pi\Omega^3R^5/550$

r radial position on rotor blade in TPP

R blade radius

S equilibrium factor, term in a_1 or A_o equation (see Appendix)

T rotor thrust force

dT differential blade element rotor thrust force, $dT \quad (dL)_a$

TPP abbreviation for tip-path plane

U resultant blade element velocity, $U = \Omega R\sqrt{v_a^2 + v_t^2 + v_r^2}$

V flight path velocity

v uniform induced velocity, $v = \dfrac{\frac{1}{2}C_T\Omega R}{(\lambda^2 + \mu^2)^{1/2}}$

V_A freestream inflow velocity ratio, $V_A = \dfrac{V\sin\alpha_R}{\Omega R}$

v_i triangular induced velocity distribution, non-dimensional coefficient

$$v_i = \frac{\frac{3}{4}C_T}{(1-\mu^2)\sqrt{(V_A - \frac{3}{4}v_i)^2 + \mu^2}}$$

v_a non-dimensional axial component of resultant blade element velocity, $v_a = \lambda - a_o\mu\cos\psi$

v_r non-dimensional radial component of resultant blade element velocity, $v_r = \mu\cos\psi$

v_t non-dimensional tangential component of resultant blade element velocity, $v_t = x + \mu\sin\psi$

W helicopter weight

x non-dimensional rotor blade radius, $x = r/R$

x_1 non-dimensional inboard blade section radius, $x_1 = r_1/R$

x_2 non-dimensional effective outboard blade section radius, $x_2 = r_2/R$

X,Y,Z equilibrium factors, terms in a_1 or A_o equation (see Appendix), also tip-path plane axes designations

ZLL abbreviation for blade element zero lift chord line

ΩR rotor blade rotational tip speed

SUMMARY

As the forward speed ratios of helicopters increase, it becomes necessary at some point to take into account the radial velocity corrections. The purpose of this study is to correct current lifting rotor theory for the effects of the radial component of resultant blade element velocity.

Force, moment, and power equations, including the effects of the radial component of the resultant velocity at the rotor blade elements, are derived. Since certain integral multipliers appear repeatedly in all equations derived, these multipliers were evaluated for flight conditions ranging from forward speed ratios of zero to one. The results of the evaluations of the integral multipliers are presented in tabular form in order to facilitate computations involved in applying the derived equations. Values in the tabular form were determined utilizing a Burroughs 220 computer.

The derived equations are set up with the assumptions of uniform induced velocity distribution. Second and higher harmonic flapping is neglected. Tip-path plane co-ordinates are used. Further, the assumption of constant chord blades is made throughout. The effects of linear blade twist are included in the equations.

A theoretical comparison is made between results obtained, using the equations derived herein and similar equations which neglect the effects of the radial component. The comparison was made for a current production-model helicopter. The comparison shows that, at a forward

speed of 193 mph, the calculated total power required increases approximately 15 per cent when the effects of the radial velocity component at the blade elements are taken into account.

Moreover, it was found that both the rotor lift and profile powers increased due to the radial velocity component. Of the two, the per cent increase in the profile power is greater.

CHAPTER I

INTRODUCTION

The majority of current helicopters are designed to fly at top forward speeds which range from 100 to 150 mph, although speeds as high as 200 mph have been obtained. Contemporary lifting rotor theory, such as that used in performance prediction, gives results that agree reasonably well with flight test data for the speed range mentioned above.

However, the mathematical labor involved in developing contemporary lifting rotor theory is considerable, and to simplify this mathematical complexity most authors have disregarded the effects of the radial component of freestream velocity acting at the blade elements (as seen in Fig. 1).

With the advent of helicopter designs which have projected flight speeds well above present ranges, the contemporary theories that neglect the effects of the radial component may become inadequate. In particular, it is believed that contemporary performance theory, taking into account only the tangential and axial components of the resultant velocity at the blade elements, will underestimate the helicopter power required, the error becoming greater at higher forward speeds.

The purpose of this paper is to correct contemporary lifting rotor theory by including the effects of the radial component. This will be done by employing certain series expansions for the blade bound vortex strength and the resultant velocity at the blade elements, as described in Reference 1.

In general, the basic assumptions of two-dimensional airfoil theory will be adhered to throughout. Compressibility will not be considered, except in the effect on the average blade element lift curve slope. Equations derived will automatically take into account the reversed flow region, and be applicable for large inflow angles. As stated earlier, applications will be made to constant chord, linearly twisted blades only.

It is the aim of this study to gain a general idea of the error involved in contemporary theory which neglects the radial component. To do this, a typical problem will be considered. Using the derived power-required equation and a similar relation from Reference 2 which neglects the radial component, the difference in the power required at a given flight condition will be computed.

CHAPTER II

THEORETICAL ANALYSIS

Blade Bound Vortex Distribution.--A typical rotor tip-path plane coordinate system is chosen, as shown in Fig. 1. The tip-path plane can be visualized as that plane described by the blade tips of an operating helicopter main-rotor system. The z-axis is perpendicular to the tip-path plane in the direction of the thrust. The x- and y-axes lie fixed in the tip-path plane with the x-axis directed downwind and the y-axis directed to starboard. Let the flight path or freestream velocity be V and the rotor angle of attack be α_R.

As is usually assumed for simplicity in making approximate rotor performance calculations, let the induced velocity, v, be uniform over the rotor disk area swept by the blades. Also, neglect second and higher harmonic rotor flapping motions, since these effects are small in most applications.

A rotating system of helicopter rotor blades describes a cone as a result of mass centrifugal forces balancing blade aerodynamic forces in the flapping mode. With respect to the tip-path plane, the described cone is of base angle, a_o, called the coning angle in helicopter terminology. The coning angle will be assumed small so that $\sin a_o \approx a_o$ and the $\cos a_o \approx 1$.

For simplicity, assume the b number of blades in the main rotor are of constant chord, c. Let the rotor angular velocity be Ω and the rotor blade radius be R. Then the relative velocity components

acting along the blade at any radial position, r, become

$$(\text{axial component}) = V \sin \alpha_R - v - (V \cos \alpha_R \cos \psi) a_o \tag{1}$$

$$(\text{tangential component}) = \Omega r + V \cos \alpha_R \sin \psi \tag{2}$$

$$(\text{radial component}) = V \cos \alpha_R \cos \psi \tag{3}$$

where, as assumed previously, a_o is small and ψ is the blade azimuth position measured from the downwind x-axis in the direction of rotation, as shown in Fig. 1.

In non-dimensional form, which will be convenient for subsequent derivations, Equations (1), (2), and (3) become

$$(\text{axial component}) \; v_a = \lambda - a_o \mu \cos \psi \tag{4}$$

$$(\text{tangential component}) \; v_t = x + \mu \sin \psi \tag{5}$$

$$(\text{radial component}) \; v_r = \mu \cos \psi \tag{6}$$

where

$$\lambda = \frac{V \sin \alpha_R - v}{\Omega R}$$

$$x = \frac{r}{R}$$

$$\mu = \frac{V \cos \alpha_R}{\Omega R}$$

As shown in Fig. 2a, the blade angle measured about the blade axis and between the zero lift chord line and the tip-path plane is denoted by

θ (positive above the tip-path plane). The blade angle, Θ, is measured with respect to a blade element perpendicular to the blade axis. Let the variation in Θ for any flight condition be represented by the trigonometric series

$$\Theta = A_o + \Theta_1 x - a_1 \sin \psi + b_1 \cos \psi \tag{7}$$

where the term $\Theta_1 x$ gives the blade linear twist; the term A_o represents a mean value of Θ per revolution at the blade root position; and rotor blade cyclic or feathering first harmonic motions are represented by $- a_1 \sin \psi + b_1 \cos \psi$.

However, from Fig. 2b it is seen that Θ is the blade angle of a blade element perpendicular to the blade feathering axis, but at an angle η with respect to the true flow plane. The resultant velocity at the blade element in the true flow plane is

$$U = \Omega R \sqrt{v_a^2 + v_t^2 + v_r^2} \tag{8}$$

where the rotor blade tip speed, ΩR , dimensionalizes the terms under the radical. Equation (8) gives the resultant velocity acting on a yawed blade section of chord c' shown in Fig. 2c. Thus, the Kutta condition at the blade trailing edge (assumed sharp) which determines the blade circulation or bound vortex strength, is actually satisfied in the true flow plane of Fig. 2c. It is significant to point out that contemporary theory, neglecting v_r , assumes the Kutta condition to be satisfied in the plane perpendicular to the blade axis shown in Fig. 2a.

In the present case where part of the flow is "frozen" to the blade

on account of the rotation of the co-ordinate system, the usual two-dimensional skewed flow approximations do not apply. For example, if the blade axis were warped to that three-dimensional curved shape which would give the same angle of attack and skew angle distribution in a rectilinear flow as the actual blade experiences in the real flow, then the bound and wake vortex configurations would be entirely different.

Hence, working in the true flow plane in which the Kutta condition is satisfied, let the true blade angle be θ' . Then, as deduced from the geometry of Fig. 2b

$$\theta' = \theta \cos \eta = \theta \frac{v_t}{\sqrt{v_t^2 + v_r^2}} \tag{9}$$

Likewise, from Fig. 2c the true inflow angle, ϕ' , measured between the tip-path plane and the rotor blade resultant velocity, U , is

$$\phi' = \tan^{-1} \frac{v_a}{\sqrt{v_t^2 + v_r^2}} \tag{10}$$

The yawed blade chord, c' , is seen from Fig. 2b to be

$$c' = c \sec \eta = c \frac{\sqrt{v_t^2 + v_r^2}}{v_t} \tag{11}$$

where c is of constant magnitude and c' changes with ψ . Note that c' is measured in the true flow plane or parallel to U .

Let the true angle of attack be denoted by α' and measured between the zero lift line and U in Fig. 2c. The true angle of attack

which determines the blade circulation is then

$$\alpha' = \theta' + \phi' \tag{12}$$

Two-dimensional airfoil theory demonstrates from Reference 3 that for local blade element lift coefficients up to the stall

$$\frac{C_l}{a} = \sin \alpha' = \sin(\theta' + \phi') \tag{13}$$

where a = slope of lift curve. For the helicopter blade airfoil for which θ' is a small angle (although the inflow angle ϕ' is not necessarily small) it follows that $\sin \theta' \approx \theta'$ and $\cos \theta' \approx 1$. Hence, upon expanding Equation (13) with the trigonometric formula for the sine of the sum of two angles the result is

$$\frac{C_l}{a} = \theta' \cos \phi' + \sin \phi' \tag{14}$$

The local blade circulation, Γ, of a blade element in the true flow plane at position $(x, \mathcal{Y})$ is deduced from theoretical and experimental aerodynamic expressions for the lift force acting perpendicular to U and given by

$$\Gamma = \frac{1}{2} c' C_l U \tag{15}$$

Finally, the expression for the rotor blade bound vortex strength about a blade section in the true flow plane at any position $(x, \mathcal{Y})$ of the tip-path plane is found by substituting Equations (8), (9), (10), (11), and (14) into Equation (15). After simple algebraic manipulations,

the result is

$$\frac{2\Gamma}{ac\Omega R} = \left(\theta + \frac{v_a}{v_t}\right)\sqrt{v_t^2 + v_r^2} \tag{16}$$

The significance of Equation (16) is twofold. First, it includes the effects of v_r which will enable the effects of the radial component to be introduced into subsequent force and power expressions. Second, Γ retains the correct sign in the reversed flow region (i.e., positive with respect to a negative U), and thus affords continuous load distributions for vibration analyses and obviates the necessity of making reverse flow corrections.

Rotor Thrust Force.--In this section an equation including the effects of v_r and representing the mean rotor thrust force per revolution will be derived. The differential lift force, dL , acting on the differential blade element, cdr , located in the tip-path plane at position (r , ψ) is obtained as

$$dL = \rho \Omega R \Gamma \sqrt{v_a^2 + v_t^2}\, dr \tag{17}$$

which is shown in Fig. 3. The tip speed, ΩR , is included in Equation (17) to dimensionalize the terms under the radical.

The resultant differential lift force can be broken into a differential axial force component, $(dL)_a$, perpendicular to the tip-path plane, and into the tangential component, $(dL)_t$, parallel to the tip-path plane. With reference to Fig. 3, the axial component can be found by resolving dL into the axial direction. Thus, from similar triangles, Equation (17)

is multiplied by $\cos \phi = v_t / \sqrt{v_a^2 + v_t^2}$ resulting in

$$(dL)_a = \rho \Omega R \Gamma v_t dr \tag{18}$$

which is positive in the plus z-direction. Likewise, the tangential component is obtained by multiplying Equation (17) by $\sin \phi = v_a / \sqrt{v_a^2 + v_t^2}$ resulting in

$$(dL)_t = - \rho \Omega R \Gamma v_a dr \tag{19}$$

assumed positive opposite the direction of rotation.

It is important to note that Equations (17), (18), and (19) have been derived with respect to the rotor normal and tangential directions. Such a treatment simplifies the derivations and does not leave out v_r since the radial component enters the equations by way of the term, Γ, given by Equation (16).

It should be remembered that the mean or steady value of the resultant rotor thrust force is defined as perpendicular to the tip-path plane. Then, since a_o has been assumed small, the differential axial blade force, $(dL)_a$, lies in the same direction as dT, the differential blade thrust force. Thus

$$(dL)_a = dT \tag{20}$$

Then from Equation (18)

$$dT = \rho \Omega R \Gamma v_t dr \tag{21}$$

Substituting Equation (16) into (21) gives

$$dT = \frac{\rho a c \Omega^2 R^2}{2} (\theta v_t + v_a) \sqrt{v_t^2 + v_r^2}\, dr \tag{22}$$

Define a non-dimensional blade force or thrust coefficient as

$$C_T = \frac{T}{\rho \pi \Omega^2 R^4} \tag{23}$$

and note that the differential of the non-dimensional blade radius, $x = r/R$, is $\frac{dr}{R} = dx$. Thus, Equation (22) can be re-written as

$$\frac{dC_T}{dx} = \frac{ac}{2\pi R} (\theta v_t + v_a) \sqrt{v_t^2 + v_r^2} \tag{24}$$

which gives the slope of the non-dimensional thrust loading along a blade at some azimuth position ψ (see Fig. 4). Note that v_r is included in Equation (24).

It is desired to obtain an expression for the average or mean rotor thrust force per revolution for any flight condition. The individual blade thrust is found by integrating Equation (24) from an inboard section, x_1, to some effective outboard section, x_2, near or at the tip. Then the average of the blade thrust for the revolution is found by integrating azimuthally from $\psi = 0$ to $\psi = 2$ and dividing by 2π. Multiplying the above by the b number of blades in the rotor system gives the desired result. Thus

$$C_T = b\left(\frac{1}{2\pi}\right) \int_0^{2\pi} \left[\int_{x_1}^{x_2} \frac{dC_T}{dx}\, dx \right] d\psi \tag{25}$$

or

$$C_T = \frac{a\sigma'}{2}\left(\frac{1}{2\pi}\right)\int_0^{2\pi}\int_{x_1}^{x_2}\left[(\theta\, v_t + v_a)\sqrt{v_t^2 + v_r^2}\right] dx\, d\psi \qquad (26)$$

where $\sigma' = bc/\pi R$ defined as the non-dimensional rotor solidity ratio

x_1 non-dimensional inboard blade limit

x_2 non-dimensional outboard blade limit

$C'_T = \int_{x_1}^{x_2} \frac{dC_T}{dx}\, dx$ rotor blade thrust force coefficient

Although Equation (26) can be readily integrated with respect to the blade radius, the result cannot, in general, be integrated in closed form with respect to blade azimuthally position ψ . However, the troublesome radical $\sqrt{v_t^2 + v_r^2}$ can be expanded in a converging series.

Substituting Equations (5) and (6) into $\sqrt{v_t^2 + v_r^2}$ and factoring out the average value for the revolution gives

$$\sqrt{v_t^2 + v_r^2} = \sqrt{x^2 + \mu^2}\sqrt{1 + 2F} \qquad (27)$$

where

$$F = \frac{\mu x \sin\psi}{x^2 + \mu^2} \qquad (28)$$

Applying the binomial expansion to the factor $\sqrt{1 + 2F}$ on the right side of Equation (27) gives

$$\sqrt{1 + 2F} = 1 + F - \frac{1}{2}F^2 + \ldots\ldots \qquad (29)$$

which Castles, in Reference 1, has shown to converge for all μ, x and ψ, except at the isolated point $x = \mu$ and $\psi = 270°$ where the dynamic pressure and lift force are essentially zero. Castles has also shown that utilizing only the three terms in the series gives accurate results in the calculation of blade loadings, as compared to exact numerical integration. Thus, substituting Equations (27) and (29) into (24) gives

$$\frac{dC_T}{dx} = \frac{ac}{2\pi R}(\theta v_t + v_a)(x^2 + u^2)^{1/2}(1 + F - \frac{1}{2}F^2) \qquad (30)$$

At this point, it is convenient to consider the equation for the thrust force acting on a blade in position ψ. Thus, substituting Equations (28) and (30) into (25) and integrating with respect to x only gives

$$C'_T = \frac{a\sigma}{2b}\left\{\left[(\lambda\mu^4 - \frac{1}{2}a_1\mu^5)\bar{\Phi}_1 + (\frac{3}{2}A_o\mu^4 + \frac{3}{16}a_1\mu^3 - \frac{1}{4}\lambda\mu^2)\bar{\Phi}_2\right.\right.$$

$$+ (2\lambda\mu^2 - \frac{3}{2}a_1\mu^3 + \frac{3}{2}\theta_1\mu^4 - \frac{1}{4}A_o\mu^2)\bar{\Phi}_3 + (\frac{5}{2}A_o\mu^2 - \frac{1}{4}\theta_1\mu^2)\bar{\Phi}_4$$

$$\left.+ (\lambda + \frac{5}{2}\theta_1\mu^2 - a_1\mu)\bar{\Phi}_5 + (A_o)\bar{\Phi}_6 + (\theta_1)\bar{\Phi}_7\right] + \sin\psi\left[(A_o\mu^5)\bar{\Phi}_1\right.$$

$$+ (\theta_1\mu^5 + \lambda\mu^3 - \frac{3}{8}A_o\mu^3 - \frac{7}{4}a_1\mu^4)\bar{\Phi}_2 + (3A_o\mu^3 + \frac{3}{8}a_1\mu^2$$

$$- \frac{3}{8}\theta_1\mu^3)\bar{\Phi}_3 + (\lambda\mu + 3\theta_1\mu^3 - \frac{11}{4}a_1\mu^2)\bar{\Phi}_4 + (2A_o\mu)\bar{\Phi}_5 + (2\theta_1\mu$$

$$\left.- a_1)\bar{\Phi}_6\right] + \cos\psi\left[(-a_o\mu^5)\bar{\Phi}_1 + (\frac{5}{4}b_1\mu^4 + \frac{1}{8}a_o\mu^3)\bar{\Phi}_2 + (-2a_o\mu^3\right.$$

$$- \frac{1}{8} b_1 \mu^2)\bar{\Phi}_3 + (\frac{9}{4} b_1 \mu^2)\bar{\Phi}_4 + (\frac{9}{4} a_o u)\bar{\Phi}_4 + (- a_o \mu)\bar{\Phi}_5 + (b_1)\bar{\Phi}_6\Big]$$

$$+ \sin 2\psi \Big[(\frac{1}{2} b_1 \mu^5)\bar{\Phi}_1 + (- \frac{1}{2} a_o \mu^4 - \frac{1}{8} b_1 \mu^3)\bar{\Phi}_2 + (\frac{3}{2} b_1 \mu^3)\bar{\Phi}_3$$

$$+ (- \frac{1}{2} a_o \mu^2)\bar{\Phi}_4 + (b_1 \mu)\bar{\Phi}_5\Big] + \cos 2\psi \Big[(\frac{1}{2} a_1 \mu^5)\bar{\Phi}_1 + (- \frac{1}{2} A_o \mu^4$$

$$- \frac{1}{4} a_1 \mu^3 + \frac{1}{4} \lambda \mu^2)\bar{\Phi}_2 + (\frac{3}{2} a_1 \mu^3 - \frac{1}{2} \theta_1 \mu^4 + \frac{1}{4} A_o \mu^2)\bar{\Phi}_3$$

$$+ (- \frac{1}{2} A_o \mu^2 + \frac{1}{4} \theta_1 \mu^2)\bar{\Phi}_4 + (a_1 \mu - \frac{1}{2} \theta_1 \mu^2)\bar{\Phi}_5\Big] + \sin 3\psi \Big[(\frac{1}{4} a_1 \mu^4$$

$$+ \frac{1}{8} A_o \mu^3)\bar{\Phi}_2 + (- \frac{1}{8} a_1 \mu^2 + \frac{1}{8} \theta_1 \mu^3)\bar{\Phi}_3 + (\frac{1}{4} a_1 \mu^2)\bar{\Phi}_4\Big]$$

$$+ \cos 3\psi \Big[(- \frac{1}{4} b_1 \mu^4 - \frac{1}{8} a_o \mu^3)\bar{\Phi}_2 + (\frac{1}{8} b_1 \mu^2)\bar{\Phi}_3 + (- \frac{1}{4} b_1 \mu^2)\bar{\Phi}_4\Big]$$

$$+ \sin 4\psi \Big[(\frac{1}{16} b_1 \mu^3)\bar{\Phi}_2\Big] + \cos 4\psi \Big[(\frac{1}{16} a_1 \mu^3)\bar{\Phi}_2\Big] \qquad (31)$$

where the multipliers $\bar{\Phi}_n$ $n = 1, 2, 3, 4, \ldots\ldots$ are representative of the general integral

$$\bar{\Phi}_n = \int_{x_1}^{x_2} \frac{x^{n-1}\, dx}{(x^2 + \mu^2)^{3/2}} \qquad (32)$$

Substituting Equation (31) into (25) and integrating with respect to ψ gives the mean or average value of the non-dimensional rotor thrust force

per revolution as

$$C_T = \frac{a\sigma}{2}\left[\left(\lambda\mu^4 - \frac{1}{2}a_1\mu^5\right)\Phi_1 + \left(\frac{3}{2}A_o\mu^4 + \frac{3}{16}a_1\mu^3 - \frac{1}{4}\lambda\mu^2\right)\Phi_2\right.$$

$$+ \left(2\lambda\mu^2 - \frac{3}{2}a_1\mu^3 + \frac{3}{2}\theta_1\mu^4 - \frac{1}{4}A_o\mu^2\right)\Phi_3 + \left(\frac{5}{2}A_o\mu^2 - \frac{1}{4}\theta_1\mu^2\right)\Phi_4$$

$$\left. + \left(\lambda + \frac{5}{2}\theta_1\mu^2 - a_1\mu\right)\Phi_5 + (A_o)\Phi_6 + (\theta_1)\Phi_7\right] \qquad (33)$$

The use of Equations (31) and (33) is facilitated since the integral multipliers, Φ_n $n = 1, 2, 3, \ldots$ are tabulated for various values of forward speed ratio, μ, in Table I. To use Table I for values of μ falling between those given, conventional interpolation techniques should be used. The unknowns a_o, A_o, a_1 and b_1 can be evaluated from expressions which will be developed in subsequent sections.

It is important to note that a simplification was afforded in obtaining Equation (33) by integrating Equation (31) with respect to ψ. Note that such terms as $\sin m\psi$ or $\cos m\psi$ reduce to zero when integrated as

$$\int_0^{2\pi} (\sin m\psi \quad \text{or} \quad \cos m\psi)\, d\psi = 0 \qquad (34)$$

where $m = 1, 2, 3, 4, \ldots$

Rotor Thrust Moment.--In this section an equation including the effects of v_r and representing the mean rotor thrust moment will be derived. Consider a rotor blade at some azimuth position, ψ, as shown in Fig. 4. Distributed along the span of this blade from x_1 to x_2 and acting in the

axial or positive z-direction is the varying aerodynamic force or thrust distribution (called the thrust loading), given by Equation (30).

Considering only the instantaneous thrust loading on the blade in position ψ, it is seen that the unbalanced force distribution will produce a moment about the blade root called the rotor blade thrust moment, M_T. Considering the thrust loading as concentrated at some non-dimensional position, x, along the blade span, then x will be the lever arm of the thrust moment.

Define a blade thrust moment coefficient as

$$C_{M_T} = \frac{M_T}{\rho \pi \Omega^2 R^5} \tag{35}$$

Thus, in non-dimensional form, the rotor blade thrust moment described above which acts instantaneously on a blade at some azimuth position, ψ, becomes

$$\int_{x_1}^{x_2} \frac{dC_{MT}}{dx}\, dx = \int_{x_1}^{x_2} \frac{dC_T}{dx}\, x\, dx \tag{36}$$

where dC_T/dx is given in Equation (30).

To get the average value of the thrust moment acting on a rotor blade, consider one complete revolution of a blade and integrate Equation (36) from $\psi = 0$ to $\psi = 2\pi$. Thus

$$C_{MT} = \left(\frac{1}{2\pi}\right) \int_0^{2\pi} \left[\int_{x_1}^{x_2} \frac{dC_T}{dx} \, x \, dx \right] d\psi \tag{37}$$

However, inspection of Equation (37) reveals that the bracketed term is nearly the same as that given in Equation (25), which was expanded as Equation (31). The x multiplier in Equation (37) is the only difference. The x multiplier has the simple effect of increasing x^{n-1} to x^n in all integral multipliers of Equation (31). The integral multipliers are given by Equation (32). Further consideration of Equation (32) shows that increasing x^{n-1} to x^n is the same as increasing $\bar{\Phi}_n$ to $\bar{\Phi}_{n+1}$.

Thus, increasing the integral multipliers in Equation (31) by one unit, substituting the result into Equation (37), and integrating with respect to ψ while noting the simplification afforded in Equation (34), the average rotor blade thrust moment is obtained as

$$C_{MT} = \frac{a\sigma}{2b} \left[\left(\lambda \mu^4 - \frac{1}{2} a_1 \mu^5 \right) \bar{\Phi}_2 + \left(\frac{3}{2} A_o \mu^4 + \frac{3}{16} a_1 \mu^3 - \frac{1}{4} \lambda \mu^2 \right) \bar{\Phi}_3 \right.$$

$$+ \left(2\lambda\mu^2 - \frac{3}{2} a_1 \mu^3 + \frac{3}{2} \theta_1 \mu^4 - \frac{1}{4} A_o \mu^2 \right) \bar{\Phi}_4 + \left(\frac{5}{2} A_o \mu^2 - \frac{1}{4} \theta_1 \mu^2 \right) \bar{\Phi}_5$$

$$\left. + \left(\lambda + \frac{5}{2} \theta_1 \mu^2 - a_1 \mu \right) \bar{\Phi}_6 + (A_o) \bar{\Phi}_7 + (\theta_1) \bar{\Phi}_8 \right] \tag{38}$$

Rotor Rolling Moment.--From Fig. 4 it can be seen that the thrust moment of an element of thrust located at non-dimensional radius x has an in-

stantaneous moment arm $x \sin \psi$ about the x-axis. Defining the x-axis as the rotor rolling axis, the resulting moment described above is termed the rotor rolling moment, M_X .

Define a rotor rolling moment coefficient as

$$C_{MX} = \frac{M_X}{\rho \pi \Omega^2 R^5} \tag{39}$$

Thus, in non-dimensional form, the rotor blade rolling moment which acts instantaneously on a blade at some azimuth position, ψ , becomes

$$\int_{x_1}^{x_2} \frac{dC_{MX}}{dx}\, dx = \int_{x_1}^{x_2} \frac{dC_T}{dx}\, x \sin \psi \, dx \tag{40}$$

To get the average value of the rotor rolling moment per revolution, follow the same procedure discussed between Equations (36) and (38). However, in this case, the multiplier on dC_T/dx is $x \sin \psi$ instead of x . This gives no trouble, since the simplification afforded by Equation (34) still applies. Furthermore, note that

$$\frac{1}{2\pi} \int_0^{2\pi} \sin^2 \psi \, d\psi = \frac{1}{2} \tag{41}$$

Thus, the average value of the rotor rolling moment for a rotor with b blades becomes

$$C_{MX} = \frac{a\sigma}{4} \left[(A_o \mu^5) \bar{\omega}_2 + \left(\lambda \mu^3 - \frac{3}{8} A_o \mu^3 + \theta \mu^5 - \frac{7}{4} a_1 \mu^4 \right) \bar{\omega}_3 + (3A_o \mu^3 \right.$$

$$+ \frac{3}{8} a_1 \mu^2 - \frac{3}{8} \theta_1 \mu^3)\bar{\alpha}_4 + (\lambda \mu + 3 \theta_1 \mu^3 - \frac{11}{4} a_1 \mu^2)\bar{\alpha}_5 + (2A_o \mu)\bar{\alpha}_6$$

$$+ (2 \theta_1 \mu - a_1)\bar{\alpha}_7\Big] \tag{42}$$

Rotor Pitching Moment.--From Fig. 4 it can be seen that the thrust moment of an element of thrust located at non-dimensional radius x has an instantaneous moment arm $x \cos \psi$ about the y-axis. Defining the y-axis as the rotor pitching axis, the resulting moment described above is termed the rotor pitching moment, M_Y .

Define a rotor pitching moment coefficient as

$$C_{MY} = \frac{M_Y}{\rho \pi \Omega^2 R^5} \tag{43}$$

Thus, in non-dimensional form, the rotor blade pitching moment which acts instantaneously on a blade at some azimuth position, ψ , becomes

$$\int_{x_1}^{x_2} \frac{dC_{MY}}{dx} dx = \int_{x_1}^{x_2} \frac{dC_T}{dx} x \cos \psi \, dx \tag{44}$$

To get the average value of the rotor pitching moment per revolution, follow the same procedure discussed between Equations (36) and (38). However, in this case, the multiplier on dC_T/dx is $x \cos \psi$ instead of x . This gives no trouble, since the simplification afforded by Equation (34) still applies. Furthermore, note that

$$\frac{1}{2\pi} \int_0^{2\pi} \cos^2 \psi \, d\psi = \frac{1}{2} \tag{45}$$

Thus, the average value of the rotor pitching moment for a rotor with b blades becomes

$$C_{MY} = \frac{a\sigma}{4}\left[(-a_o\mu^5)\bar{\omega}_2 + (\frac{5}{4}b_1\mu^4 + \frac{1}{8}a_o\mu^3)\bar{\omega}_3 + (-2a_o\mu^3 - \frac{1}{8}b_1\mu^2)\bar{\omega}_4 + (\frac{9}{4}b_1\mu^2)\bar{\omega}_5 + (-a_o\mu)\bar{\omega}_6 + (b_1)\bar{\omega}_7\right] \quad (46)$$

In summary, Equations (33), (38), (42), and (46) give four equations with which to solve for the four unknown rotor parameters a_o, A_o, a_1, and b_1. This will be done in the next section.

Equilibrium Solution for Rotor Parameters.--Consider a centrally hinged, freely-flapping rotor system. For equilibrium steady state flight $C_{MX} = 0$. Furthermore, if it is assumed that rotor forces are in equilibrium and that there is no work done by the motion along the flight path of unbalanced residual forces, then Equations (33) and (42) can be solved simutaneously for A_o and a_1. Thus,

$$a_1 = \frac{(ZS - YN)}{(XS - MY)} \quad (47)$$

$$A_o = \frac{(MZS - NYM - NZS - N^2Y)}{(XS^2 - MYS)} \quad (48)$$

where the equilibrium factors X, Y, Z, M, N and S are all independent of a_o, A_o, a_1, b_1, and may be found in the Appendix. It was convenient to solve for a_1 and A_o (independent of a_o and b_1) first, since the following equation for a_o will contain a_1, A_o, and be independent of b_1.

The steady part of the blade inertia moment about the flapping hinge is largely the centrifugal moment since second and higher harmonic flapping, with respect to the tip-path plane, have been neglected. The centrifugal moment, from Reference 2, is given by $a_o I_1 \Omega^2$ where I_1 is the blade mass moment of inertia about the flapping hinge. Thus, for equilibrium, the thrust moment must balance the centrifugal moment about the flapping hinge as

$$M_T \approx a_o I_1 \Omega^2 \tag{49}$$

or non-dimensionally

$$C_{MT} = \frac{a_o I_1 \Omega^2}{\rho \pi \Omega^2 R^5} = a_o \left[\frac{I_1}{\rho \pi R^5} \right] \tag{50}$$

Thus, substituting Equation (38) into (50) and solving for a_o gives

$$a_o = K \left[\left(\lambda \mu^4 - \tfrac{1}{2} a_1 \mu^5 \right) \bar{\Phi}_2 + \left(\tfrac{3}{2} A_o \mu^4 + \tfrac{3}{16} a_1 \mu^3 - \tfrac{1}{4} \lambda \mu^2 \right) \bar{\Phi}_3 + \left(2 \lambda \mu^2 \right.\right.$$

$$\left. - \tfrac{3}{2} \theta_1 \mu^4 - \tfrac{1}{4} A_o \mu^2 \right) \bar{\Phi}_4 + \left(\tfrac{5}{2} A_o \mu^2 - \tfrac{1}{4} \theta_1 \mu^2 \right) \bar{\Phi}_5 + \left(\lambda + \tfrac{5}{2} \theta_1 \mu^2 \right.$$

$$\left.\left. - a_1 \mu \right) \bar{\Phi}_6 + \left(A_o \right) \bar{\Phi}_7 + \left(\theta_1 \right) \bar{\Phi}_8 \right] \tag{51}$$

where $K = \left[\frac{\rho \pi R^5 \sigma a}{2 b I_1} \right]$ rotor blade inertia factor.

Also, considering equilibrium steady state flight, $C_{MY} = 0$. Thus, setting Equation (46) equal to zero and solving for b_1 gives

$$b_1 = \frac{1}{\left[-\bar{\Phi}_7 + \frac{1}{8}\mu^2\bar{\Phi}_4 - \frac{5}{4}\mu^4\bar{\Phi}_3 - \frac{9}{4}\mu^2\bar{\Phi}_5\right]} \left[(-a_o\mu^5)\bar{\Phi}_2\right.$$

$$\left. + (\tfrac{1}{8}a_o\mu^3)\bar{\Phi}_3 + (-2a_o\mu^3)\bar{\Phi}_4 + (-a_o\mu)\bar{\Phi}_6\right] \qquad (52)$$

Note that b_1 is only a function of a_o, and that a_o is obtained from Equation (51) which, in turn, is evaluated from Equations (47) and (48). Thus, a convenient sequence has been established for evaluating a_1, A_o, a_o, and then b_1.

Equations (47), (48), (51), and (52) take into account the effects of the radial component and are needed to evaluate the preceding rotor thrust and moment expressions. They will also be needed to solve the helicopter power required expressions, which will be derived in the next section. In evaluating Equations (47), (48), (51), and (52) it should be remembered that the integral multipliers, $\bar{\Phi}_{n = 1, 2, 3, 4, \ldots}$, are given in Table I.

Rotor Lift Power.--In this section an equation including the effects of v_r and representing that part of the total helicopter power required due to blade lift forces will be derived. The power required for the lift on a blade element is equal to the component of lift multiplied by its associated velocity component. Thus, the differential lift power required as seen from Fig. 3 is

$$(dP)_{RL} = (dL)_t\ (\Omega R v_t) \qquad (53)$$

where ΩR dimensionalizes v_t. However, Equation (53) cannot be conveniently integrated, and must be expressed in another form. From the

conservation of energy, the energy per unit time input to the blade, $(dL)_t\ (\Omega R v_t)$, is equivalent to the energy output by the blade, $(dL)_a\ (\Omega R v_a)$. With this in mind Equation (53) may be re-written as

$$(dP)_{RL} = -\ (dL)_a\ (\Omega R v_a) \tag{54}$$

in which the minus sign is placed since $(dP)_{RL}$ must be positive when there is a downflow through the rotor represented by a negative v_a .

Substituting Equation (18) into (54) gives

$$(dP)_{RL} = -\ \rho\, \Omega^2 R^2 \Gamma v_a v_t dr \tag{55}$$

Substituting Equation (16) into (55) gives

$$(dP)_{RL} = -\ \frac{\rho\, ac\, \Omega^3 R^5}{2}\, v_a(\theta\, v_t + v_a)\sqrt{v^2_t + v^2_r}\, dr \tag{56}$$

Define a non-dimensional power coefficient as

$$C_P = \frac{P}{\rho\, \pi\, \Omega^3 R^5} \tag{57}$$

Finally, substituting Equations (27), (29), and (57) into (56), while noting that $\frac{dr}{R} = dx$, gives

$$\frac{(dC_P)_{RL}}{dx} = -\ \frac{ac}{2\pi R}\, v_a(\theta\, v_t + v_a)(x^2 + \mu^2)^{1/2}(1 + F - \tfrac{1}{2} F^2) \tag{58}$$

which represents the lift power loading required due to the variation in spanwise inflow angle and lift.

However, it is desired to obtain an expression for the average lift

power required of the whole rotor for any flight condition. This is found by integrating Equation (58) from a blade inboard section, x_1, to some outboard section, x_2, and then taking the average value for the revolution by integrating azimuthally from $\psi = 0$ to $\psi = 2\pi$. Multiplying by the b number of blades in the rotor system gives the desired result. Then

$$C_{PRL} = \frac{a\sigma}{2}\left(\frac{1}{2\pi}\right)\int_0^{2\pi}\int_{x_1}^{x_2} v_a(\theta\, v_t + v_a)(x^2 + \mu^2)^{1/2}\left(1 + F - \tfrac{1}{2}F^2\right)dx\, d\psi \quad (59)$$

To carry out the integration indicated in Equation (59) substitute Equations (4), (5), (7), and (28) into the integrand, expand the result, and integrate term by term. The same simplification afforded by Equation (34) applies and the integral multipliers are given by Equation (32). The result is written in the following tabular form (Equation (60)) where the coefficients in the boxes must each be multiplied by its corresponding row and column head. After summing all individual products of a boxed coefficient, its row head, and its column head, the resulting lengthy expression represents the average value of C_{PRL} per revolution.

$$\frac{4C_{PRL}}{a\sigma} =$$

	1	μ	μ^2	μ^3	μ^4	μ^5	μ^6
Φ_1					$-2\lambda^2$	$a_1\lambda$	$-a_o^2$
Φ_2			$\frac{1}{2}\lambda^2$	$-\frac{3}{8}a_1\lambda$	$-3A_o\lambda + \frac{1}{8}a_o^2$	$-b_1a_o$	$\frac{1}{4}b_1a_o$
Φ_3			$-4\lambda^2 + \frac{1}{2}A_o\lambda + \frac{1}{2}\theta_1\lambda$	$2a_1\lambda - \frac{1}{8}b_1a_o$	$-2\theta_1\lambda - 2a_o^2$	$-\theta_1\lambda$	
Φ_4			$-5A_o\lambda$	$\frac{7}{4}b_1a_o$			
Φ_5	$-2\lambda^2$	$2a_1\lambda$	$-5\theta_1\lambda - a_o^2$				
Φ_6	$-2A_o\lambda$	b_1a_o					
Φ_7	$-2\theta_1\lambda$						

(60)

Rotor Profile Power.--It is now desired to derive an expression for the rotor profile power required, taking into account the effects of the radial velocity component. In the non-viscous, ideal fluid the only power was that required to overcome rotor blade lift forces. In an incompressible viscous fluid, additional power must be applied to the rotor in order to overcome skin friction and pressure drag forces. However, the usual subsonic operation below the stall will be assumed. Under this conventional

assumption the viscous resistance to blade motion is mostly due to fluid shearing forces in laminar and turbulent boundary layers (defined as skin friction drag).

Hence, the differential profile drag force, dD, acting on the differential blade element, cdr, at radius, r, and azimuth position, ψ, and acting in the direction of the resultant velocity, U, is in large part skin friction and is written

$$dD = C_{do} \frac{1}{2} \rho U^2 cdr \tag{61}$$

where U is given in Equation (8).

Equation (61) may be split into three components by writing U^2 as $U(U)$ and substituting for U in parenthesis its value from Equation (8). Thus, for the axial component of Equation (61) set $v_r = v_t = 0$. The result is

$$(dD)_a = C_{do} \frac{1}{2} \rho U(\Omega R v_a) cdr \tag{62}$$

Likewise, radial and tangential components become respectively

$$(dD)_r = C_{do} \frac{1}{2} \rho U(\Omega R v_r) cdr \tag{63}$$

$$(dD)_t = C_{do} \frac{1}{2} \rho U(\Omega R v_t) cdr \tag{64}$$

Thus, the energy per second or the differential profile power dissipated at the blade element cdr due to component skin friction drag forces may be expressed by the relation

$$(dP)_{RD} = \left(\sqrt{(dD)_2^2 + (dD)_t^2 + (dD)_r^2}\right)U\, dr \tag{65}$$

Substituting Equations (62), (63), and (64) into (65) then yields

$$(dP)_{RD} = \frac{1}{2}\, \rho\, U \Omega^2 R^2 c\, C_{do}(v_a^2 + v_t^2 + v_r^2)dr \tag{66}$$

Equation (66) still leaves much to be desired, since it cannot be conveniently integrated as it stands. Further development of the terms U and C_{do} must be made.

Following essentially the same procedure in the series approximation of Equation (27), the term, U, of Equation (66), contains the troublesome radical, $\sqrt{v_a^2 + v_t^2 + v_r^2}$, which may be expanded by the binomial theorem. Then, since a_o is small for conventional designs, the resulting expansion from Reference 1 becomes

$$\frac{U}{\Omega R} = \sqrt{x^2 + \mu^2 + \lambda^2}\,(1 + G - \tfrac{1}{2} G^2) \tag{67}$$

where

$$G \approx \frac{\mu\, x \sin \gamma}{x^2 + \mu^2 + \lambda^2} \tag{68}$$

and Equation (67) is in non-dimensional form. For $\lambda \neq 0$ the series in Equation (67) converges for all values of μ, x, γ and λ. Castles, in Reference 1, explains that the convergence is similar to, but somewhat more rapid than, that of Equation (27), and will give accurate results when incorporated into the profile power expression.

The local blade profile drag coefficient, C_{do}, will now be con-

sidered. Payne, in Reference 4, explains that for smooth, thin airfoils in incompressible, viscous flow operating at subsonic speeds below the stall, the blade element profile drag is essentially skin friction and is given by the non-dimensional relation

$$C_{do} = 2\, C_f(1 + C_\ell^{\,2}) \tag{69}$$

where C_f is an average skin friction drag coefficient which, for smooth surfaces and the usual Reynolds Numbers (i.e., 1.0×10^6 to 4.0×10^6), is of the order $C_f \approx 0.004$, as seen in Reference 4.

To relate Equation (69) to the case at hand, first, Equation (15) is re-written

$$C_\ell = \frac{2\Gamma}{c'U} \tag{70}$$

Substituting Equations (8), (11), and (16) into (70) yields

$$C_\ell = \frac{a(\theta\, v_t + v_a)}{\sqrt{v_a^{\,2} + v_t^{\,2} + v_r^{\,2}}} \tag{71}$$

With the expressions for U and C_{do} thus obtained, Equation (66) can be put into integrable form. Hence, substituting Equations (67), (69), and (71) into (66), non-dimensionalizing with Equation (57), and remembering that $dr = R\,dx$, the result is

$$\frac{(dC_P)_{RD}}{dx} = \frac{cC_f}{\pi R}\left[(v_a^{\,2} + v_t^{\,2} + v_r^{\,2}) + a^2(\theta\, v_t + v_a)^2\right] \cdot$$

$$\cdot\, (x^2 + u^2 + \lambda^2)^{1/2}\left(1 + G - \tfrac{1}{2} G^2\right) \tag{72}$$

which gives the spanwise differential profile power loading required to mainly overcome skin friction. Note that v_r is included in Equation (72).

However, it is desired to obtain an expression for the average profile power required of the whole rotor for any flight condition. This is found by integrating Equation (72) from a blade inboard section, x_1, to some outboard section, x_2, and then taking the average value for the revolution by integrating azimuthally from $\psi = 0$ to $\psi = 2$. Multiplying the result by the b number of blades in the rotor system gives the desired result. Thus

$$C_{PRD} = \sigma C_f \left(\frac{1}{2\pi}\right) \int_0^2 \int_{x_1}^{x_2} \left[(v_a^2 + v_t^2 + v_r^2) + a^2 (\theta v_t + v_a)^2 \right] \cdot (x^2 + \mu^2 + \lambda^2)^{1/2} \left(1 + G - \tfrac{1}{2} G^2\right) dx \, d\psi \qquad (73)$$

To carry out the integration indicated in Equation (73), substitute Equations (4), (5), (6), (7), and (68) into the integrand, expand the result, and integrate term by term. The same simplification afforded by Equation (34) applies and the integral multipliers are given by

$$\mathfrak{F}'_n = \int_{x_1}^{x_2} \frac{x^{n-1} \, dx}{\left[x^2 + (\mu^2 + \lambda^2)\right]^{3/2}} \qquad (74)$$

where for high speed forward flight $\mu^2 >> \lambda^2$. Hence, as a first approximation λ^2 can be dropped from Equation (74). With the previous assumption Equation (74) becomes identical to Equation (32) (i.e., $\Phi_n \equiv \Phi'_n$). Thus, Table I may be used in evaluating C_{PRD}.

The result of the integration in Equation (73) is written for convenience in the following tabular form (Equation (75)) where the coefficients in the boxes must each be multiplied by its corresponding row and column head. After summing all individual products of a boxed coefficient, its row head, and its column head, the resulting lengthy expression represents the average value of C_{PRD} per revolution.

$$\frac{2\,C_{PRD}}{a^2 \sigma\, C_f} =$$

	1	μ	μ^2	μ^3	μ^4	μ^5	μ^6
$\mathfrak{G}'_1$	$\frac{2\lambda^6}{a^2} + \frac{\lambda^4 a_o^2}{a^2}$	$-2a_1\lambda^5 + a_o\lambda^4$	$\frac{6\lambda^4}{a^2} + A_o^2\lambda^4 + \frac{3}{4}a_1^2\lambda^4 + \frac{2a_o^2}{a^2}$	$2a_o^2\lambda^2 - 4a_1\lambda^3$	$\frac{6\lambda^2}{a^2} + 2A_o\lambda^2 + a_1^2\lambda^2 + \frac{a_o^2}{a^2}$	$-2\lambda a_1$	$\frac{2}{a^2} + A_o^2 + \frac{3}{4}a_1^2 + a_o^2$
$\mathfrak{G}'_2$	$2A_o^2\lambda^4 + 4A_o\lambda^5$	$2a_1^2\lambda^4 - 4A_o a_1\lambda^4 + 2b_1 a_o\lambda^4$	$2A_o\theta_1\lambda^4 + 4A_o^2\lambda^2 + 10A_o\lambda^3$	$4a_1^2\lambda^2 - \frac{19}{2}A_o a_1\lambda^2 - \frac{9}{2}b_1 a_o\lambda^2$	$2A_o^2 + 6A_o\lambda + 4A_o\theta_1\lambda^2$	$2a_1^2 - \frac{11}{2}A_o a_1 - \frac{5}{2}b_1 a_o$	$2A_o\theta_1$
$\mathfrak{G}'_3$	$\frac{6\lambda^4}{a^2} + 4\theta_1\lambda^5 + a_1^2\lambda^4 + b_1^2\lambda^4 + \frac{2\lambda^2 a_o^2}{a^2}$	$-2\theta_1 a_1\lambda^2 - 6\lambda^3 a_1 - 4\theta_1 a_1\lambda^4$	$\frac{27}{2a^2} + \frac{17a_o^2}{8a^2} + \frac{1}{2}\lambda^2 + 4A_o^2\lambda^2 + \frac{11}{4}a_1^2\lambda^2 + 2a_o^2 + \theta_1^2\lambda^4 + 10\theta_1\lambda^3 + \frac{5}{2}b_1^2\lambda^3$	$-2\theta_1 a_1 - \frac{21}{4}\lambda a_1 - \frac{19}{2}\theta_1 a_1\lambda^2$	$\frac{15}{2a^2} + \frac{29}{8}A_o^2 + \frac{23}{16}a_1^2 + 2a_o^2 + \frac{1}{8} + 6\theta_1\lambda + \frac{23}{16}b_1^2 + 2\theta_1^2\lambda^2$	$-\frac{11}{2}\theta_1 a_1$	θ_1^2
$\mathfrak{G}'_4$	$4A_o^2\lambda^2 + 8A_o\lambda^3 + 4A_o\theta_1\lambda^4$	$4a_1^2\lambda^2 - 10A_o a_1\lambda^2 - 4b_1 a_o\lambda^2$	$4A_o^2 + 9A_o\lambda + 16A_o\theta_1\lambda^2$	$\frac{13}{4}a_1^2 - \frac{23}{2}A_o a_1 - \frac{19}{4}b_1 a_o$	$\frac{45}{4}A_o\theta_1$		

(continued)

	1	μ	μ^2	μ^3	μ^4	μ^5	μ^6
Φ'_5	$\frac{6\lambda^2}{a^2} + 8\theta_1\lambda^3$ $+2a_1\lambda^2+2b_1^2\frac{\lambda^2}{a^2}$ $+ 2\theta_1^2\lambda^4 + \frac{a_o}{a^2}$	$-4\lambda a_1 - 6a_1\theta_1\lambda^2$	$\frac{15}{2a^2} + \frac{5}{2}A_o^2 + a_o^2$ $+ 9\theta_1\lambda + 2a_1$ $+ \frac{19}{8}b_1^2 + 8\theta_1^2\lambda^2$	$- 6\theta_1 a_1$	$\frac{45}{8}\theta_1^2$		
Φ'_6	$4A_o\lambda + 8A_o\theta_1\lambda^2$	$2a_1^2 - 6A_o a_1$ $-2b_1 a_o$	$13 A_o\theta_1$				
Φ'_7	$\frac{2}{a^2}+2A_o^2+4\theta_1\lambda$ $+a_1^2+b_1^2+4\theta_1^2\lambda^2$	$- 6\theta_1 a_1$	$6\theta_1^2$				
Φ'_8	$4 A_o\theta_1$						
Φ'_9	$2\theta_1^2$						

(concluded)

(75)

Total Helicopter Power Required.--The last step in this analysis is to develop an equation for the total helicopter power required which incorporates the effects of the radial velocity component. The complete expression is constructed by combining Equations (60) and (75), since total helicopter power required is equal to the lift power plus the profile power. Thus, in non-dimensional form, the equation would be

$$C_{PRT} = \frac{a\sigma}{4} \sum_{i=0}^{i=6} \sum_{n=1}^{n=7} C_{in} \mu^i \bar{\Phi}_n + \frac{a^2 \sigma C_f}{2} \sum_{i=0}^{i=6} \sum_{n=1}^{n=9} C_{in} \mu^i \bar{\Phi}'_n$$

$$= C_{PRL} + C_{PRD} \qquad (76)$$

where the summations represent the expansion of the tabular forms of Equations (60) and (75). That is, each boxed term, C_{in}, in the i-th column and the n-th row is to be multiplied by its corresponding i-th column head, μ^i, its n-th row head, $\bar{\Phi}_n$ or $\bar{\Phi}'_n$; then all individual products are summed.

Inspection of Equations (60) and (75) will show that Equation (76) is quite lengthy. Furthermore, Equation (76) would have to be modified for compressibility and rotor blade stall under the most general assumptions of this analysis.

However, a useful expression can be obtained for C_{PRT} for speeds below that at which tip stall on the retreating blade occurs. Thus, the following order analysis is performed.

Assume that terms containing powers of μ higher than the fourth are small and can be dropped. Next, consider fifth and higher order products of μ, λ and the blade angles are small and can be dropped.

Lastly, drop all terms containing a_o and b_1 since experience shows that such terms tend to cancel one another. Thus, the result of expanding Equation (76) becomes

$$C_{PRT} = \frac{a\sigma}{4}\left[(\tfrac{1}{2}\lambda^2\mu^2)\bar{\Phi}_2 + (\tfrac{1}{2}A_o\lambda\mu^2 - 4\lambda^2\mu^2 + \tfrac{1}{2}\theta_1\lambda\mu^2)\bar{\Phi}_3\right.$$

$$+ (-5A_o\lambda\mu^2)\bar{\Phi}_4 + (-2\lambda^2 + 2a_1\lambda\mu - 5\theta_1\lambda\mu^2)\bar{\Phi}_5$$

$$\left. + (-2A_o\lambda)\bar{\Phi}_6 + (-2\theta_1\lambda)\bar{\Phi}_7\right] + \frac{a^2\sigma C_f}{2}\left[(\frac{6\lambda^4}{a^2} + \frac{27}{2a^2}\mu^2\right.$$

$$- \tfrac{1}{2}\lambda^2\mu^2 + \frac{15}{2a^2}\mu^4 - \tfrac{1}{8}\mu^4)\bar{\Phi}'_3 + (4A_o^2\lambda^2 + 8A_o\lambda^3 + 4A_o^2\mu^2$$

$$+ 9A_o\lambda\mu^2)\bar{\Phi}'_4 + (\frac{6\lambda^2}{a^2} + 8\theta_1\lambda^3 + 2a_1\lambda^2 - 4\lambda a_1\mu + \frac{15}{2a^2}\mu^2$$

$$+ \tfrac{5}{2}A_o^2\mu^2 + 9\theta_1\lambda\mu^2 + 2a_1\mu^2)\bar{\Phi}'_5 + (4A_o\lambda + 8A_o\theta_1\lambda^2 + 2a_1^2\mu$$

$$- 6A_oa_1\mu + 13A_o\theta_1\mu^2)\bar{\Phi}'_6 + (\frac{2}{a^2} + 2A_o^2 + 4\theta_1\lambda + a_1^2 + 4\theta_1^2\lambda^2$$

$$\left. - 6\theta_1a_1\mu + 6\theta_1^2\mu^2)\bar{\Phi}'_7 + (4A_o\theta_1)\bar{\Phi}'_8 + (2\theta_1^2)\bar{\Phi}'_9\right] \qquad (77)$$

Since Equation (77) is approximate, it is convenient to assume that $\bar{\Phi}_n \approx \bar{\Phi}'_n$. Thus, Table I can be used to facilitate the evaluation of Equation (77). The unknowns, A_o and a_1, are obtained from Equations (47) and (48), respectively.

To complete a tip-path plane analysis an equation for the rotor X-force must be included. Such an equation is obtained from Reference 2 and is given in the Appendix.

CHAPTER III

APPLICATION

Sample Problem.--It is desired to compare two computed values of the power required for a given flight condition. The first value computed using Equation (77) (which accounts for v_r) will be compared with a similar value computed using the equation in the Appendix (which neglects v_r). The helicopter considered will be the Bell YHU-1B and the following given data were obtained from Reference 5.

Given Data:

$V = 193$ mph	$b = 2$
$f = 20$ ft^2	$\theta_1 = 10°$
$W = 6000$ lbs	$a_o = 4°$
$\sigma = 0.051$	$\Omega = 323$ rpm
$R = 22$ ft	

Approximate Procedure: Where not specifically stated, terms used in the following procedure are defined in the List of Symbols.

1) Assume that $T = W$ as a first approximation in calculating C_T.
2) As a first approximation neglect F_x and compute α_R from $\alpha_R \approx \tan^{-1} D_F/W$.
3) Use Equations (47) and (48) in calculating a_1 and A_o to be used in Equation (77).
4) Since Equation (77) is based on a uniform induced velocity, v_i, use the expression for λ obtained from Reference 6.

5) The equation for C_{PR} in the Appendix is based on a triangular induced velocity, v_i. Thus, use the theory of Reference 2 to compute v_i.

6) As a first approximation assume $\lambda \approx 0$ in the integral multipliers given by Equation (74). Thus, Equation (74) becomes identical to Equation (32) (i.e., $\Phi_n \equiv \Phi'_n$). Thus, for the corresponding μ at $V = 193$ mph interpolate values for the Φ_n's from Fig. 5.

7) Values obtained above are included in Table II, from which the equations for C_{PRT} and C_{PR} are evaluated.

8) Neglect transmission and tail rotor losses.

Approximate Results: From the evaluation of Equations (77) for C_{PRT} and the equation for C_{PR} in the Appendix, it was found, dimensionally, that for the given flight condition the total power required became

(with v_r)	$P_R = 1550$ shp
(without v_r)	$P_R = 1290$ shp

where it was found that

$$C_{PRL} = 0.000456$$

$$C_{PRD} = 0.000120$$

CHAPTER IV

CONCLUSIONS

Force, moment, and power equations, including the effects of the radial component of the resultant velocity at the rotor blade elements, were derived. Integral multipliers contained in the derived equations were tabulated. With data obtained on the Bell YHU-1B, the power required was calculated at a high forward speed flight condition, both with the equations derived and similar ones neglecting the radial velocity component. A resulting comparison shows that the effects of the radial component on the power required are as follows:

1) For the Bell YHU-1B the calculated total power required will be increased about 15 per cent when the effects of the radial velocity component at the blade elements are taken into account.
2) Of the total power required for the YHU-1B, that part due to lift will be increased about 10 per cent due to the effects of the radial velocity component.
3) Of the total power required for the YHU-1B, that part due to profile drag will be increased about 20 per cent due to the effects of the radial velocity component.
4) In general, it is concluded from this investigation that theoretical prediction of the helicopter power required will underestimate the actual power required at high forward speeds when the effects of the radial velocity component are neglected.

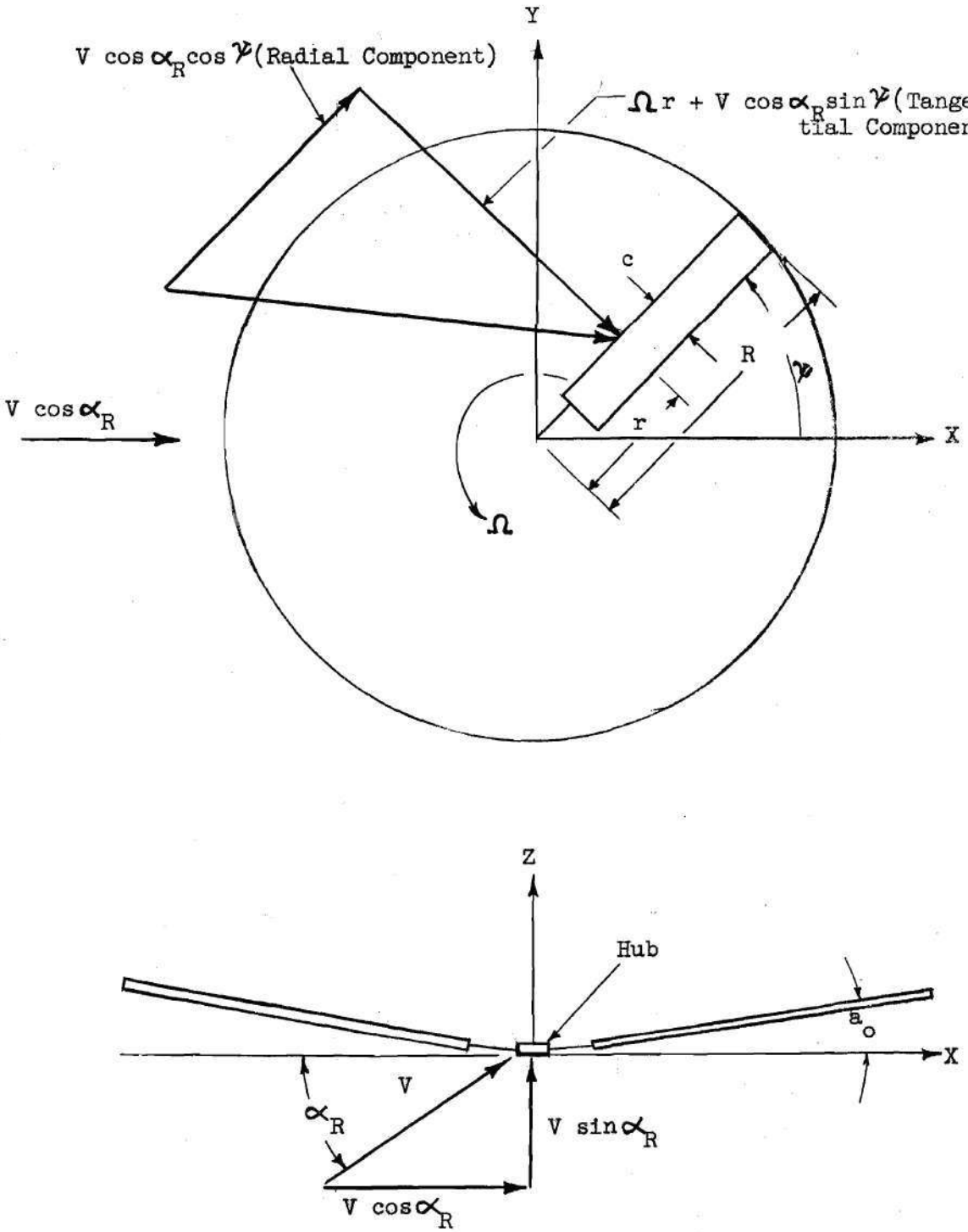

Fig. 1 Freestream and Relative Velocity Components in Rotor Tip-Path Plane Co-ordinate System (all velocities, lengths, and angles shown positive)

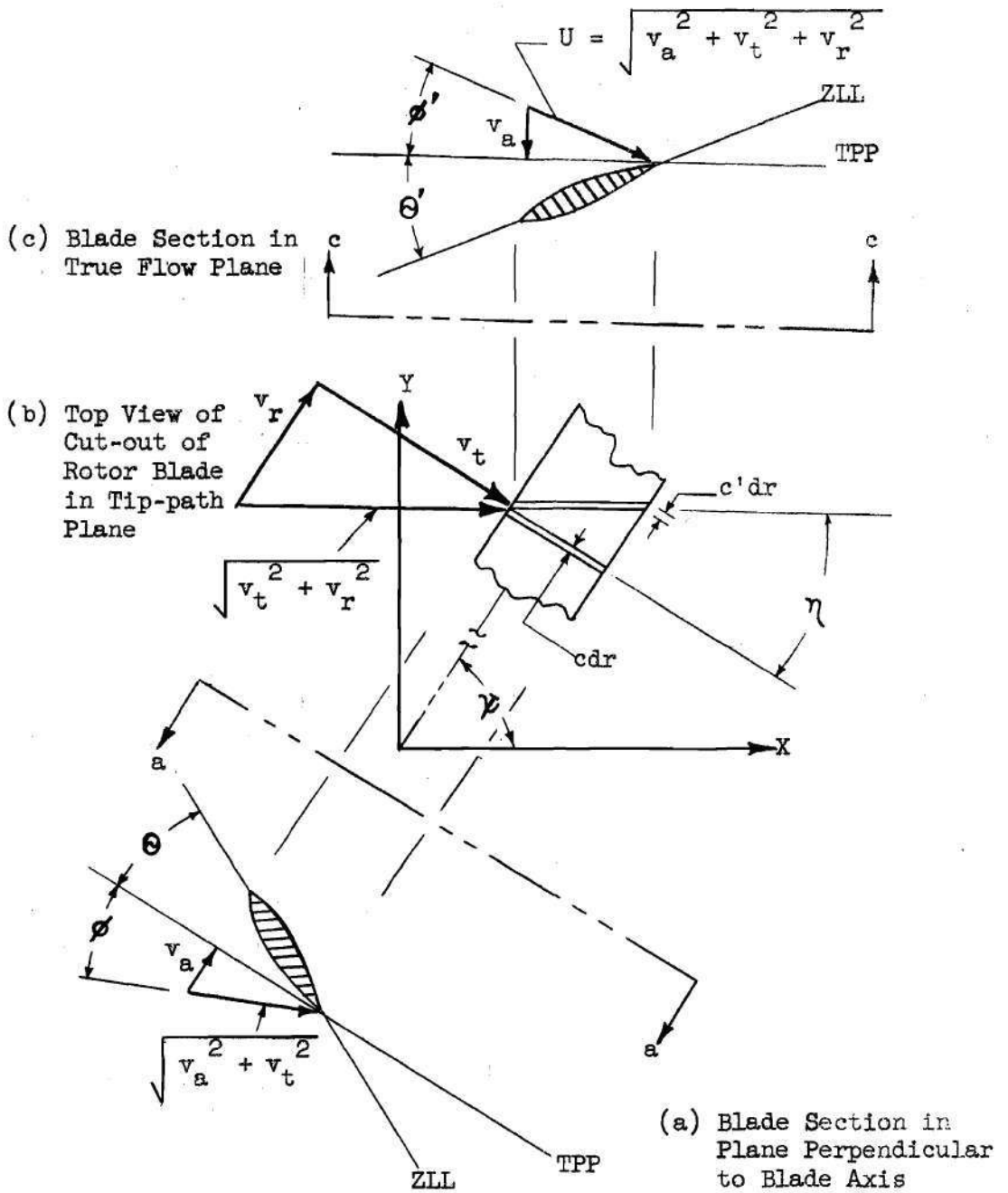

Fig. 2 Three View of Flow Planes, Velocity Components and Flow Angles (all shown positive)

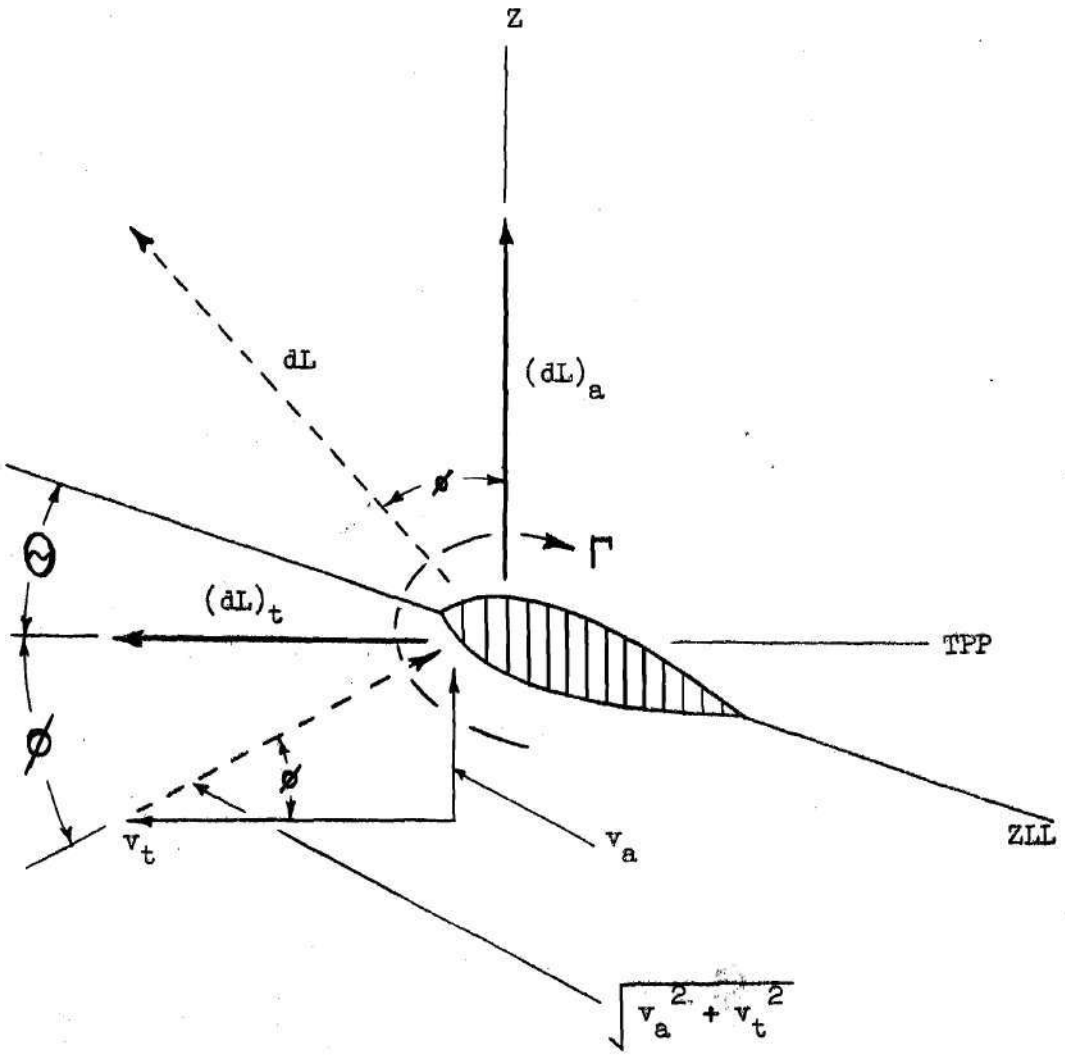

Fig. 3 Lift Force Components in Plane Perpendicular to Blade Axis

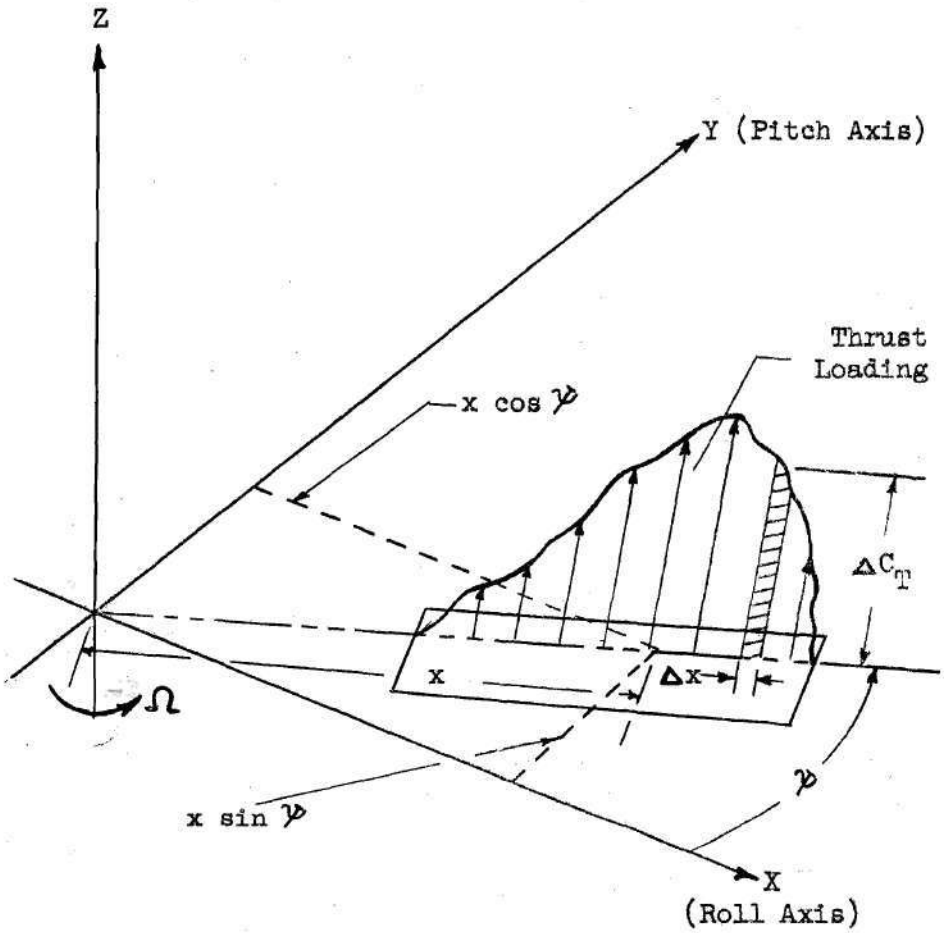

Fig. 4 Blade Loading and Moment Diagram

Table 1

Integral Multipliers for Various Helicopter

Forward Speed Ratios

μ	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8	Φ_9
0.0	14.1020	5.2614	1.8476	0.8904	0.4820	0.3210	0.2381	0.1745	0.1401
0.1	10.0300	3.4460	1.4230	0.7170	0.4287	0.2906	0.2144	0.1672	0.1355
0.2	6.8070	2.2525	1.1280	0.6065	0.3788	0.2637	0.1975	0.1555	0.1267
0.3	4.4510	1.7880	0.8639	0.4938	0.3217	0.2299	0.1750	0.1392	0.1142
0.4	2.9820	1.2820	0.6605	0.3967	0.2674	0.1954	0.1509	0.1212	0.1001
0.5	2.0690	0.9406	0.5089	0.3176	0.2198	0.1635	0.1278	0.1035	0.0861
0.6	1.4830	0.7043	0.3961	0.2545	0.1798	0.1357	0.1071	0.0874	0.0730
0.7	1.0940	0.5376	0.3115	0.2047	0.1470	0.1121	0.0893	0.0733	0.0615
0.8	0.8264	0.4173	0.2475	0.1656	0.1204	0.0927	0.0743	0.0613	0.0516
0.9	0.6372	0.3289	0.1987	0.1348	0.0990	0.0768	0.0619	0.0512	0.0433
1.0	0.5001	0.2627	0.1611	0.1105	0.0819	0.0639	0.0517	0.0429	0.0364

Table 2

Values Used in Example Problem

μ	0.367
V (fps)	284
D_F (lbf)	1920
α_R(o)	- 16.7
V_A	- 0.110
v_i	0.00680
v	0.00390
λ	- 0.114
C_T	0.00301
θ_1 (radian)	- 0.174
A_o (radian)	0.127
a_1 (radian)	0.118
Φ_2	1.434
Φ_3	0.721
Φ_4	0.326
Φ_5	0.284
Φ_6	0.206
Φ_7	0.158
Φ_8	0.127
Φ_9	0.104

APPENDIX

Equilibrium Factors.--The following terms are found in Equations (47) and (48):

$$M = \left(\frac{3}{16}\mu^3 \Phi_3 - \frac{1}{2}\mu^5 \Phi_2 - \frac{3}{2}\mu^3 \Phi_4 - \mu \Phi_6\right)$$

$$N = \left(\lambda\mu^4 \Phi_2 - \frac{1}{4}\lambda\mu^2 \Phi_3 + 2\lambda\mu^2 \Phi_4 + \frac{3}{2}\theta_1\mu^4 \Phi_4 - \theta_1\mu^2 \Phi_5 + \lambda \Phi_6 + \frac{5}{2}\theta\mu^2 \Phi_6 + \theta_1 \Phi_8 - \frac{2C_T}{a\sigma}\right)$$

$$S = \left(\frac{1}{4}\mu^2 \Phi_3 - \Phi_6 - \frac{3}{2}\mu^4 \Phi_2 - \frac{5}{2}\mu^2 \Phi_4\right)$$

$$X = \left(\Phi_7 + \frac{7}{4}u^4 \Phi_3 - \frac{3}{8}u^2 \Phi_4 + \frac{11}{4}u^2 \Phi_5\right)$$

$$Y = \left(\mu^5 \Phi_2 - \frac{3}{8}\mu^3 \Phi_3 + 3\mu^3 \Phi_4 + 2\mu \Phi_6\right)$$

$$Z = \left(\lambda\mu^3 \Phi_3 + \theta_1\mu^5 \Phi_3 - \frac{3}{8}\theta_1\mu^3 \Phi_4 + \lambda\mu \Phi_5 + 3\theta_1\mu^3 \Phi_5 + 2\mu\theta_1 \Phi_7\right)$$

X-Force Equation.--The following equation was obtained from Reference 2. The significance of the X-force is that it is needed for an accurate determination of α_R. The X-force is the downwind component of inplane blade element lift and profile forces. Non-dimensionally

$$C_x = \left(\frac{C_T}{1 - \mu^2}\right)\left(2V_A - \frac{10}{3}v_i\right)\mu + \frac{\delta_o\sigma\mu}{2} - \frac{16\epsilon}{\sigma}\left(\frac{C_T}{1 - \mu^2}\right)^2\mu$$

where the above equation neglects effects of reversed flow and the radial velocity component; however, nearly all other assumptions of the preceding analysis apply. Thus, as a first approximation the above equation may be used along with the other equations in the analysis.

Power Equation.--For comparison purposes, as in the example problem, the following equation from Reference 2 is included. This equation is similar to Equation (77) with one exception; namely, the following equation neglects the effects of the radial velocity component and reversed flow.

$$C_{PR} = \left(\frac{C_T}{1 - \mu^2}\right)\left[\frac{3}{4} v_i\left(1 + \frac{4}{3}\mu^2\right) - V_A\right] - \frac{a\sigma V_A v_i}{48} - \frac{a\sigma a_o^2 \mu^2}{72} + \frac{1}{8}\delta_o\sigma\,(1$$

$$+ V_A^2 - \frac{4}{3} V_A v_i + \frac{1}{2} v_i^2 + \mu^2) + \frac{9}{2}\epsilon\frac{C_T^2}{\sigma}\frac{(1 + \frac{8}{9}\mu^2)}{(1 - \mu^2)} + \frac{a^2\epsilon\sigma}{8}\left(\frac{1}{4} V_A^2\right.$$

$$\left. + \frac{1}{9} a_o^2 \mu^2\right)$$

where v_i is the triangular induced velocity coefficient.

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